Vacuum Tube Electronics at Ultra-high Frequencies *

By F. B. LLEWELLYN

Vacuum tube electronics are analyzed when the time of flight of the electrons is taken into account. The analysis starts with a known current, which in general consists of direct-current value plus a number of alternating-current components. The velocities of the electrons are associated with corresponding current components, and from these velocities the potential differences are computed, so that the final result may be expressed in the form of an impedance.

Applications of the general analysis are made to diodes, triodes with negative grid, and to triodes with positive grid and either negative or positive plate which constitute the Barkhausen type of ultra-high-frequency oscillator. A wave-length range extending from infinity down to only a few centimeters is considered, and it is shown that even in the low-frequency range certain slight modifications should be made in our usual analysis of the negative grid triode.

Oscillation conditions for positive grid triodes are indicated, and a brief discussion of the general assumptions made in the theory is appended.

I. Foreword

THE art of producing, detecting, and modulating ultra-high-frequency electric oscillations has reached the same state of development which was attained in early work on lower frequency oscillations when experiment had outstripped theory. The experimenters were able to produce oscillations by using vacuum tubes, but were not able to explain why. They were able to make improvements by the long and tedious process of cut and try, but did not have the powerful tools of theoretical analysis at their command. In particular, the advantage of the theoretical attack may be illustrated by the rapid advance in technique which followed the theoretical concept of the internal cathode-plate impedance of three-element vacuum tubes. The work of van der Bijl and Nichols showed that for purposes of circuit analysis this path could be replaced by a fictitious generator of voltage, μe_g, having an internal impedance whose magnitude is given by the reciprocal of the slope of the static $V_p - I_p$ characteristic. Development of commercially reliable vacuum tube circuits began forthwith. In a similar, yet less complicated manner, the internal network of two-element tubes may be replaced by an equivalent resistance when relatively low frequencies only are considered.

In these concepts where the vacuum tube is replaced by its equiva-

* Presented in brief summary before U. R. S. I., Washington, D. C., April, 1932. *Proc. I. R. E.*, Vol. 21, No. 11, November, 1933.

lent network impedance, one outstanding feature is exemplified: namely, the separation of the alternating- and direct-current components. The equivalent networks are applicable to the alternating-current fundamental component of the current and differ widely from the direct-current characteristics. A complete realization of the importance of this separation will be of advantage in the later steps where extension of the classical theory to the case of ultra-high-frequency currents is described.

For a short time after the original introduction of the equivalent network of the tube, affairs progressed smoothly. Soon, however, frequencies were increased and a new complication arose. The difficulty was attributable to the interelectrode capacities existing between the various elements of the vacuum tube. The original attempts to take this into account were based on the viewpoint that the tube network should be complete in itself and separate from the external circuit network to which it was attached. Correct results, of course, were obtained by this method but later developments showed the advantage of considering the equivalent network of the complete circuit, including both tube and external impedances in a single network. For instance, by grouping the combination of grid-cathode capacity with whatever external impedance was connected between these two electrodes, a great simplification occurred. This step also has its analogy in the development of ultra-high-frequency relations.

As time went on, higher and higher frequencies were desired, and they were produced by the same kind of vacuum tubes operating in the same kind of circuits, although refinements in circuit and tube design allowed the technique to be improved to the point where oscillations of the order of 70 to 80 megacycles were obtainable with fair efficiency. When the frequency was increased still further, it was found that extension of the same kind of refinements was unavailing in maintaining the efficiency and mode of operation of the higher frequency oscillations at the level which had previously been secured. Ultimately, the three-electrode tube regenerative oscillator ceases to function as a power generator in the neighborhood of 100 megacycles for the more usual types of transmitting tubes. When this point was reached, the external circuit had not yet shrunk up to zero proportions and neither had its losses become sufficiently high to account altogether for the failure of the tube to produce oscillations. From this point on, the old-time cut-and-try methods were employed and marked improvements were secured. In fact, low power tubes have been made which operate at wave-lengths of the order of 50 to 100 centimeters with fair stability, although quite low efficiency.

In the meantime, the production of ultra-high-frequency oscillations had been progressing in a somewhat different direction. The discovery, about 1920, by Barkhausen that oscillations of less than 100 centimeters wave-length could be secured in a tube having a symmetrical structure, when the grid was operated at a fairly high positive potential, while the plate was approximately at the cathode potential, started experiments on what was thought to be an altogether different mode of oscillation. Workers by the score have extended both the experimental technique and the theory of production of this newer type of oscillation. However, one of the results which an analysis of ultra-high-frequency electronics illustrates is that the electron type of oscillator is merely another example of the same kind of oscillation which was produced in the old-time so-called regenerative circuits.

For the purpose of extending the theory of electronics within vacuum tubes to frequencies where the time of transit of the electrons becomes comparable with the oscillation period, it is important at the outset to select an idealized picture which is simple enough to allow exact mathematical relations to be written. At the same time, the picture must be capable of adaptation to practical circuits without undue violence to the mathematics. An example of this kind of adaptation is illustrated by the classical calculation of the amplification factor μ, which was accomplished by consideration of the force of the electrostatic field existing near the cathode in the absence of space charge even though tubes were never operated under this condition. In a like manner, such violations of the ideal must, of necessity, be made in ultra-high-frequency analysis but their practical validity lies in so choosing them that the quantitative error introduced is less than the expected precision of measurement. It becomes, therefore, of the utmost importance to state clearly the transitions which occur between results obtained for the idealized case to which the mathematics is strictly applicable and the practical circuits where the assumptions and approximations are made to conform with operating conditions.

A start has already been made on the problem of developing such a generally valid system of electronics. This was done by Benham [1] who considers a special case comprising two parallel-plane electrodes, one of which is an emitter and the other a collector, when conditions at the emitter are restricted by the assumption that the electrons are emitted with neither initial velocity nor acceleration. This work of Benham's has the utmost importance in a general electronic theory

[1] W. E. Benham, "Theory of the Internal Action of Thermionic Systems at Moderately High Frequencies," Part I, *Phil. Mag.*, p. 641; March (1928); Part II, *Phil. Mag.*, Vol. 11, p. 457; February (1931).

and, in fact, the means of extending his theory exists primarily in the selection of much more general boundary conditions than were assumed by him. It will, therefore, result that some repetition of Benham's work will appear in the following pages. However, in view of the new state of the theory and the importance of accurate foundations for it, this repetition is advantageous rather than otherwise.

With these preliminary remarks in mind, the next step is the selection of the idealized starting point for a mathematical analysis. Exactly as was done by Benham we take two parallel planes of infinite extent, one of which is held at a positive potential V with respect to the other, and between the two electrons are free to move under the influence of the existing fields. The next step in the idealization constitutes the separation of alternating- and direct-current components not only of current and potential, but also of electron velocity, charge density, and electric intensity. With this separation, the restriction that the direct-current component of the electron velocity and acceleration is zero at the negative plane may be made while leaving us free to select much more general boundary conditions for the alternating-current component. It is true that the more general conditions now proposed will not fit the original physical picture where the negative plane consists of a thermionic emitter. Nevertheless the extension is of importance since it allows application to be made to the wide number of physical cases where "virtual cathodes" are formed. One such example is the convergence of electrons toward a plate maintained at cathode potential while a grid operating at a high positive potential with respect to both is interposed between them. In a stricter mathematical sense, the broader boundary conditions come about because of the fact that the general equations containing all components are separable into a system of equations, one for each component, and that the boundary conditions for the different equations of the system are independent of each other.

The concept of an alternating-current velocity component requires a few words of explanation. In the absence of all alternating-current components, electrons leave the cathode with zero velocity and acceleration and move across to the anode with constantly increasing velocity under the well-known classical laws. This velocity constitutes the direct-current velocity component. When the alternating-current components are introduced, there will be a fluctuation in velocity superposed on the direct-current value, and the alternating-current component need not be zero at a virtual cathode. This separation of components will come about naturally in the course of the mathematical analysis which follows, but since the interpretation of the

equations is of paramount importance, a few words of explanation and repetition will be necessary.

II. Fundamental Relations

For the development of the fundamental relations existing between the two parallel planes, we have the classical equations of the electromagnetic theory which may be set down in the following form:

$$\left.\begin{aligned} E &= -\frac{\partial V}{\partial x}, \\ \frac{\partial E}{\partial x} &= 4\pi P, \\ J &= PU + \frac{1}{4\pi}\frac{\partial E}{\partial t}, \end{aligned}\right\} \tag{1}$$

where E is the electric intensity, V the potential, P the charge density, J the total current density consisting of conduction and displacement components, and U is the charge velocity. These equations apply to frequencies such that the time which would be taken by an electromagnetic wave in traveling between the two planes is inappreciable when compared with the period of any alternating-current frequency considered. Ordinarily this limitation will become of importance only at frequencies higher even than those in the centimeter wave-length range where the time of electron transit is of great importance, although the time of passage of an electromagnetic wave is still negligibly small.

An electron situated between the two parallel plates will be acted upon by a force which determines its acceleration. The resulting velocity is a function both of the distance, x, from the cathode and the time, t, so that in terms of partial derivatives, the equation expressing the relation between the force and acceleration is

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = \frac{e}{m}E. \tag{2}$$

From (1) and (2) may readily be obtained

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 U = 4\pi\frac{e}{m}J. \tag{3}$$

In this equation we have a relation between the velocity and the total current density. The advantage of this form of equation for a starting point lies in the fact that the total current density J is not a function

of x. This comes about because of the plane shape and parallel disposition of the electrodes, and the fact that current always flows in closed paths. Thus, while the current between the two planes may be a function of time, it is not a function of x.

The separation of alternating- and direct-current components may now be made. We write

$$J = J_0 + J_1 + J_2 + \cdots \tag{4}$$

with corresponding

$$\left.\begin{aligned} U &= U_0 + U_1 + U_2 + \cdots, \\ V &= V_0 + V_1 + V_2 + \cdots, \end{aligned}\right\} \tag{5}$$

where the quantities with the zero subscript are dependent on x, only, those with subscript 1 are dependent to first order of small quantities upon time, those with subscript 2 are dependent to second order, and so forth. As a result of this separation in accord with the order of dependents upon time, (3) may be split up into a system of equations, the first of which expresses the relation between U_0, J_0, and x and does not involve time. This is the relation governing the direct-current components. The second equation of the system involves the relation between U_1, J_1, x, and time, and contains U_0 which was determined by the first equation. Likewise, the third equation contains U_2, U_1, J_2, x, and t. Since the series given by (4) and (5) are convergent so that, in general, the terms with higher order subscripts are smaller than those with lower subscripts, we may consider that, at least for small values of alternating-current components, the total fundamental frequency component is given by the terms with unity subscript.

The first two equations of the system are as follows:

$$U_0 \frac{\partial}{\partial x}\left(U_0 \frac{\partial U_0}{\partial x} \right) = 4\pi \frac{e}{m} J_0, \tag{6}$$

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} \right)\left(\frac{\partial U_1}{\partial t} + U_0 \frac{\partial U_1}{\partial x} + U_1 \frac{\partial U_0}{\partial x} \right) + U_1 \frac{\partial}{\partial x}\left(U_0 \frac{\partial U_0}{\partial x} \right) = 4\pi \frac{e}{m} J_1. \tag{7}$$

In the solution of (6), the boundary conditions are restricted so that when x is zero, the velocity and acceleration both are zero. These restrictions mean that initial velocities are neglected, and that complete space charge is assumed. Thus the solution for U_0 is

$$U_0 = \alpha x^{2/3}, \tag{8}$$

where

$$\alpha = \left(18\pi \frac{e}{m} J_0 \right)^{1/3}. \tag{9}$$

The solution of (7) is more complicated. We assign a particular value to J_1, namely, $J_1 = A \sin pt$ and find the corresponding value of U_1. To do this, it is convenient to change the variable x to a new variable ξ, which will be called the transit angle. This new variable is equal to the product of the angular frequency p and the time τ which it would take an electron moving with velocity U_0 to reach the point x and is given as follows:

$$\xi = p\tau = \frac{3p}{\alpha} x^{1/3}. \tag{10}$$

Upon changing the dependent variable from U_1 to ω, where $U_1 = \omega/\xi$, we find from (7)

$$\left(\frac{\partial}{\partial t} + p \frac{\partial}{\partial \xi} \right)^2 \omega = \xi\beta \sin pt, \tag{11}$$

where

$$\beta = 4\pi \frac{e}{m} A.$$

This has the solution

$$U_1 = -\frac{\beta}{p^2} \left[\sin pt + \frac{1}{\xi} \cos pt + F_1(\xi - pt) + \frac{1}{\xi} F_2(\xi - pt) \right]. \tag{12}$$

This equation contains two arbitrary functions of $(\xi - pt)$ which must be evaluated by the boundary conditions selected for U_1. Thus the boundary conditions for the alternating-current component make their first appearance.

From the form of (7) which is linear in U_1, it is evident that U_1 must be a sinusoidal function of time having an angular frequency p in order to correspond with the form of J_1. It follows, then, that the most general form which can be assumed for the steady state functions F_1 and F_2 is as follows:

$$\left. \begin{aligned} F_1(\xi - pt) &= a \sin (\xi - pt) + b \cos (\xi - pt) \\ F_2(\xi - pt) &= c \sin (\xi - pt) + d \cos (\xi - pt) \end{aligned} \right\}. \tag{13}$$

Now for the boundary conditions. As pointed out, there is no mathematical necessity for the boundary conditions imposed upon U_1 to correspond with those which were imposed upon U_0. At an actual cathode consisting of an electron emitting surface it would be appropriate to assume that the initial velocities are in no way dependent upon the current, but we shall have to deal not only with actual

cathodes, but also with virtual [2] cathodes where the assumption of zero alternating-current velocity and acceleration is unwarranted. Such a virtual cathode might occur, for instance, between a grid operated at a positive direct-current potential and a plate nearly at cathode potential. If enough electrons came through the mesh of the grid to depress the potential until it became practically zero at some point in the space between grid and plate, the direct-current boundary conditions of zero velocity and acceleration of electrons would be fulfilled at that point. The general equations for the alternating current will therefore apply when the origin is taken at the point of direct-current potential minimum which forms the virtual cathode, and when all of the electrons which are emitted by the actual cathode pass by the virtual cathode and reach the plate. In the event that some of the electrons are turned back at the virtual cathode and move again toward the grid, as indeed they all do when the plate is at a negative potential, a change in the form of the general equation is necessary, and will be described in the sections dealing particularly with positive grid triodes. This change, however, affects merely the form of the equations and not the physical arguments underlying the selection of boundary conditions, which are the same whether all the electrons reach the plate or whether some or all of them turn back toward the grid.

If the alternating-current velocity is determined by small variations in grid potential, let us say, it is evident that no additional assumptions save the requirement that the velocity must not become infinite may be made concerning its value at the virtual cathode. Consequently, a quite general set of boundary conditions will suffice to determine the quantities, a, b, c, d, which appear in (13) and thus completely determine U_1.

Since there are two arbitrary functions in (12), two boundary conditions will be needed. Further inspection shows that the stipulation that the alternating-current velocity be finite at the origin is sufficient to furnish one of these boundary conditions. For the other, a knowledge of the value of the alternating-current velocity at any point between the two reference planes is sufficient. Thus, if at a particular value of ξ, say ξ_1, we know that U_1 is equal to $M \sin pt + N \cos pt$, we have enough information to calculate its value at all other points between the two planes. For example, the two reference planes might be the grid and plate of a positive grid triode. In this event, the alternating-current velocity at the grid could be calculated at the grid plane by means of conditions between there and the cathode.

[2] E. W. B. Gill, "A Space-Charge Effect," *Phil. Mag.*, Vol. 49, p. 993 (1925).

In mathematical form the two boundary conditions may be set forth as follows:
when,

$$\xi = 0, \qquad U_1 \text{ must be finite,} \tag{14}$$

$$\xi = \xi_1, \qquad U_1 = M \sin pt + N \cos pt. \tag{15}$$

From (12) and (13) these result in the values:

$$c = 0, \qquad d = -2,$$

$$a = \frac{p^2}{\beta}(M \cos \xi_1 - N \sin \xi_1) + \cos \xi_1 - \frac{2}{\xi^1} \sin \xi_1, \tag{16}$$

$$b = \frac{2}{\xi_1}(1 - \cos \xi_1) - \sin \xi_1 - \frac{p^2}{\beta}(M \sin \xi_1 + N \cos \xi_1). \tag{17}$$

Thence from (12) we have for the alternating-current velocity, in general,

$$\begin{aligned} U_1 = {} & (M + iN)(\cos \xi_1 + i \sin \xi_1)(\cos \xi - i \sin \xi) \\ & + \frac{\beta}{p^2}\left[\left\{\left(\cos \xi_1 - \frac{2}{\xi_1}\sin \xi_1\right) - i\left(\frac{2}{\xi_1} - \frac{2}{\xi_1}\cos \xi_1 - \sin \xi_1\right)\right\}\right. \\ & \left.(\cos \xi - i \sin \xi) - \left(1 - \frac{2}{\xi}\sin \xi\right) - i\frac{2}{\xi}(1 - \cos \xi)\right], \end{aligned} \tag{18}$$

where, in accord with engineering practice, complex notation is employed, so that sin pt has been replaced by e^{ipt} and cos pt has been replaced by ie^{ipt}, where $i = \sqrt{-1}$.

The first step in the derivation of fundamental relations has now been achieved. The alternating-current velocity at any point between the two planes has been expressed in terms of the alternating-current velocity, $M + iN$, existing at a definite value of x, say x_1, corresponding to the transit angle ξ_1.

The next step is a determination of the potentials corresponding to the velocities U_0 and U_1, respectively. Thus from (1) and (2)

$$-\frac{e}{m}\frac{\partial V}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \tag{19}$$

and then with the separation of components as given by (5)

$$-\frac{e}{m}\frac{\partial V_0}{\partial x} = U_0\frac{\partial U_0}{\partial x}, \tag{20}$$

$$-\frac{e}{m}\frac{\partial V_1}{\partial x} = \frac{\partial U_1}{\partial t} + \frac{\partial}{\partial x}(U_0 U_1). \tag{21}$$

The solution of (20) is

$$V_0 = -\frac{m}{2e} U_0^2 = -\frac{m}{2e} \alpha^2 x^{4/3}, \tag{22}$$

which is the well-known classical relation between the potential, the current, and the position between two parallel planes where complete space charge exists. The complete space-charge condition is postulated by the boundary conditions selected for U_0 and the implications involved are discussed by I. Langmuir and Karl T. Compton.[3]

The alternating-current component of the potential is obtained by integration of (21) as follows:

$$-\frac{e}{m} V_1 = \frac{\partial}{\partial t} \int U_1 \partial x + U_0 U_1 + f(t), \tag{23}$$

whence, from (18), and in complex notation

$$\begin{aligned} V_1 = &-\frac{2m}{e} \frac{\alpha^3}{9p^2} (M + iN)(\cos \xi_1 + i \sin \xi_1)[(\xi \sin \xi + \cos \xi) \\ &+ i(\xi \cos \xi - \sin \xi)] \\ &- \frac{2m\alpha^3\beta}{e9p^4} \Bigg[\left\{ \left(\cos \xi_1 - \frac{2}{\xi_1} \sin \xi_1 \right) - i \left(\frac{2}{\xi_1} - \frac{2}{\xi_1} \cos \xi_1 - \sin \xi_1 \right) \right\} \\ &\quad [(\xi \sin \xi + \cos \xi) + i(\xi \cos \xi - \sin \xi)] \\ &- \cos \xi - i(\xi + \tfrac{1}{6}\xi^3 - \sin \xi) \Bigg] + \text{constant}. \end{aligned} \tag{24}$$

With the attainment of (24), the fundamental relation between the alternating-current component J_1 and the alternating-current potential V_1 in the idealized parallel plate diode has been secured. In a more general sense the equation is applicable between any two fictitious parallel planes where one is located at an origin where the boundary conditions for U_0 are satisfied; namely, that the direct-current components of the velocity and acceleration are zero, and the value of the alternating-current velocity at a point, x_1, corresponding to the transit angle, ξ_1, is given by $M \sin pt + N \cos pt$, or by $M + iN$ in complex notation.

Equation (24) contains an additive constant which always appears in potential calculations. This constant disappears when the potential difference is computed. For instance, suppose the potential difference between planes where ξ has the values ξ and ξ', respectively, is desired.

[3] I. Langmuir and Karl T. Compton, "Electrical Discharges in Gases"—Part II, *Rev. Mod. Phys.*, Vol. 3, p. 191; April (1931).

We have

$$V_1 = f(\xi) + \text{constant},$$

$$V_1' = f(\xi') + \text{constant},$$

so that

$$V_1 - V_1' = f(\xi) - f(\xi'). \tag{24-a}$$

Since the potential difference is always required rather than the absolute potential, (24-a) gives the means for applying (24) to actual problems.

III. Application to Diodes

In the application of the fundamental relations to diodes where the thermionic emitter forms the plane located at the origin and the anode coincides with the other plane, the boundary condition is that U_1 shall be zero at the cathode. This means that both M and N are zero and that ξ_1 is also zero. The resulting forms taken by (18) and (24-a), respectively, are as follows:

$$U_1 = -\frac{\beta}{p^2}\left[\left(1 + \cos\xi - \frac{2}{\xi}\sin\xi\right) + i\left(\frac{2}{\xi} - \sin\xi - \frac{2}{\xi}\cos\xi\right)\right], \tag{25}$$

$$V_1 - V_1' = \frac{2m\alpha^3\beta}{e9p^4}\left[(2\cos\xi + \xi\sin\xi - 2) + i(\xi + \tfrac{1}{6}\xi^3 - 2\sin\xi + \xi\cos\xi)\right]. \tag{26}$$

These two equations are identical with those obtained by Benham,[1] and graphs are given in Figs. 1 and 2 showing their variation as a function of the transit angle ξ. In particular, the equivalent impedance between unit areas of the two parallel planes may be found from (26). It must be remembered that the current, A, was assumed positive when directed away from the origin. Hence, we may write

$$Z = -\frac{V_1 - V_1'}{A}. \tag{27}$$

Moreover, the coefficient outside the square brackets in the equation may be expressed more simply when it is realized that the low-frequency internal resistance of a diode is given by the expression

$$r_0 = -\frac{\partial V_0}{\partial J_0}, \tag{28}$$

the minus sign again appearing because of the assumed current direction. Consequently, under the condition of complete space charge,

we have from (22)

$$\frac{2m\alpha^3\beta}{e9p^4} = \frac{12r_0A}{\xi^4}. \tag{29}$$

In addition to the graphs in Figs. 1 and 2 showing the real and imaginary components of impedance and velocity, the graphs shown in Figs. 3 and 4 give their respective magnitudes and phase angles.

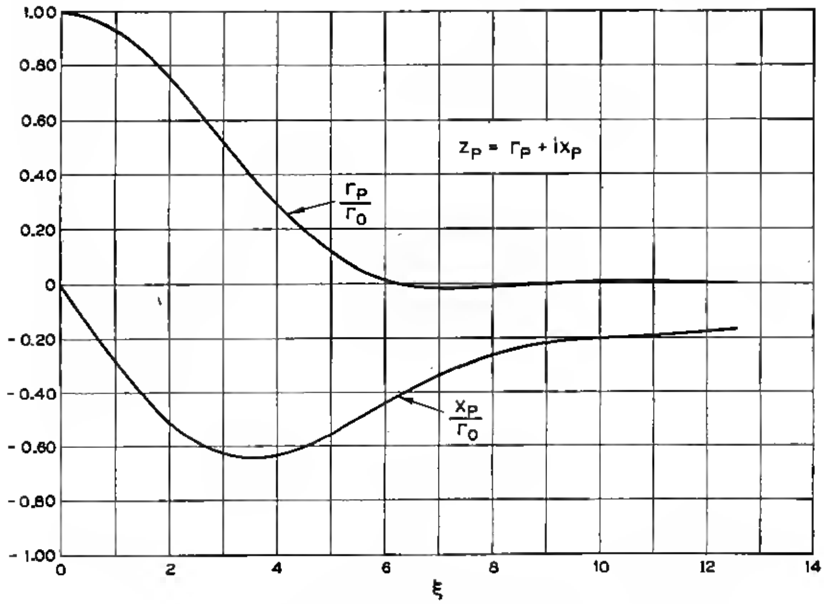

Fig. 1—Plate impedance of diodes or of negative grid triodes as a function of electron transit angle.

The impedance charts show a negative resistance for diodes in the neighborhood of a transit angle, ξ, of 7 radians. The possibility of securing oscillations in this region has been discussed by Benham, so that only a few additional remarks will be made here.

The magnitude of the ratio of reactance to resistance is about 15 when the transit angle is 7 radians. This means that oscillation conditions require an external circuit having a larger ratio of reactance to resistance. On account of the high value of reactance required, a tuned circuit or Lecher-wire system is needed, which would have to operate near an antiresonance point in order to supply the high reactance value. But the resistance component of the external circuit impedance is large at frequencies in the neighborhood of the tuning point, so that the ratio of reactance to resistance is small. Calcula-

tions show that the possibility of securing external circuits having low enough losses to meet the oscillation requirements of most of the diodes which are at present available is not very favorable. The large radio-frequency loss in the filamentary cathodes with which many tubes are supplied is an additional obstacle to be overcome before satisfactory ultra-high-frequency operation of diodes can be expected.

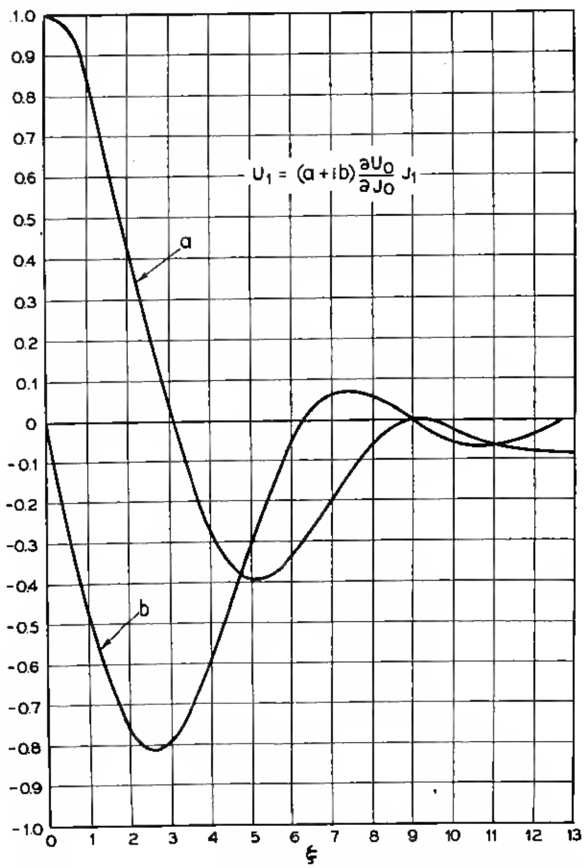

Fig. 2—Electron velocity fluctuation in diodes versus transit angle.

IV. Triodes with Negative Grid and Positive Plate

In the application of the fundamental relations to triodes operating with the grid at a negative potential, the problem becomes more complicated because of the several current paths which exist within the tube. Moreover, the direct-current potential distribution is disturbed in a radical way by the presence of the negative grid. In fact, the

negative grid triode in some respects offers greater theoretical difficulty than does the positive grid triode, which is treated in the next section. However, because of the greater ease in the interpretation of the results in terms which have become familiar through years of use, the negative grid triode is treated first.

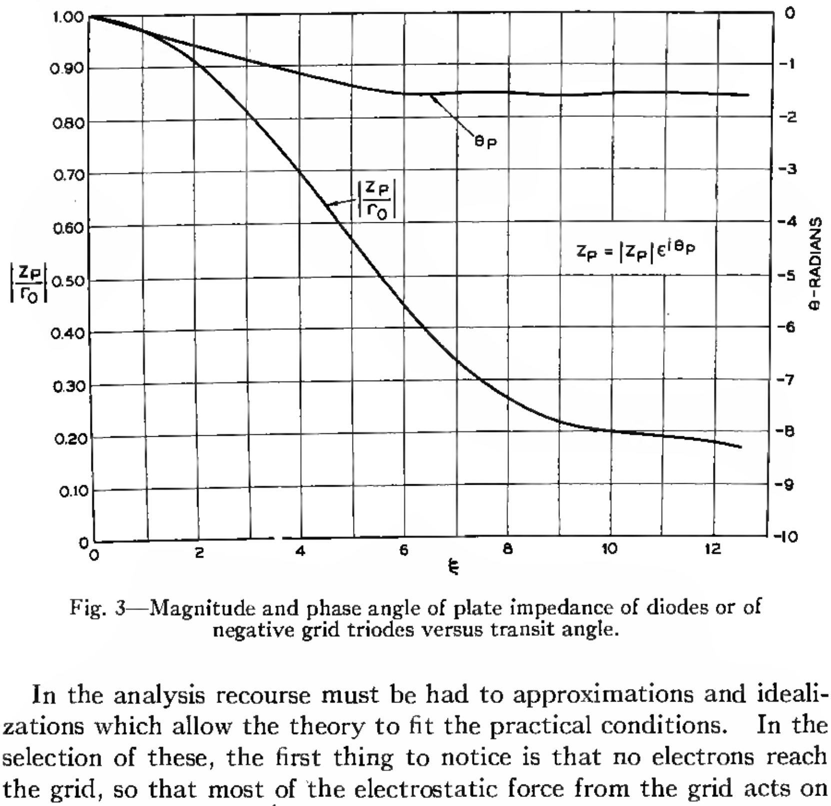

Fig. 3—Magnitude and phase angle of plate impedance of diodes or of negative grid triodes versus transit angle.

In the analysis recourse must be had to approximations and idealizations which allow the theory to fit the practical conditions. In the selection of these, the first thing to notice is that no electrons reach the grid, so that most of the electrostatic force from the grid acts on electrons quite near the cathode, where the charge density is very great. The most prominent effect of a change in grid potential will thus be a change in the velocity of electrons at a point quite near the cathode. It will thus be appropriate to assume as a starting point that the alternating-current velocity at a point x_1, located quite near the cathode is directly proportional to the alternating-current grid potential, V_g, so that we may write,

when $$\xi = \xi_1,$$

$$U_1 = (M + iN) = kV_g. \tag{30}$$

In any event, this relation may be justified if the factor of proportionality, k, be allowed to assume complex values, and ξ_1 is not taken too near the origin. Actually, the electron-free space surrounding the grid wires, and the fact that the electric intensity at a point midway between any two of the wires is directed perpendicularly to the plane of the grid, gives us more confidence in extending the approximation, so that k will be regarded as real, and ξ_1 will be taken very small.

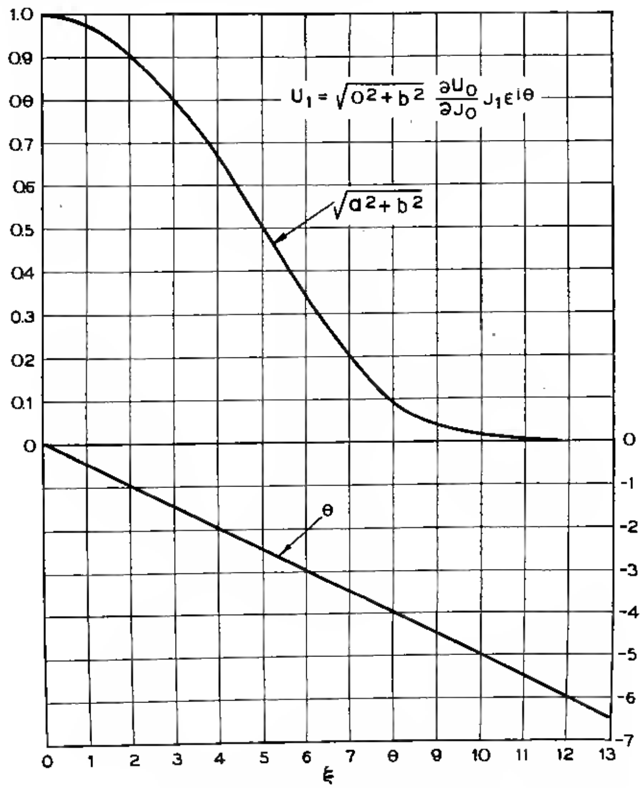

Fig. 4—Magnitude and phase angle of electron velocity fluctuation in diodes versus transit angle.

Equation (24) may, therefore, be applied under the conditions that $\xi_1 \to 0$, and gives the following for the potential difference between plate and cathode:

$$V_p = -\frac{12r_0A}{\xi^4}\Big[(\xi \sin \xi + 2\cos \xi - 2) + i(\xi + \tfrac{1}{6}\xi^3 - 2\sin \xi + \xi \cos \xi)$$
$$-(M+iN)\frac{p^2}{\beta}[(\xi \sin \xi + \cos \xi - 1) - i(\sin \xi - \xi \cos \xi)]\Big]. \quad (31)$$

This equation may be written in condensed form with the aid of (30)

$$V_p = J_1(r + ix) - V_g(\mu + i\nu), \tag{32}$$

where

$$\left.\begin{aligned} r &= -\frac{12r_0}{\xi^4}(\xi \sin \xi + 2 \cos \xi - 2), \\ x &= -\frac{12r_0}{\xi^4}(\xi + \tfrac{1}{6}\xi^3 - 2 \sin \xi + \xi \cos \xi), \\ \mu &= \frac{2\mu_0}{\xi^2}(\xi \sin \xi + \cos \xi - 1), \\ \nu &= \frac{2\mu_0}{\xi^2}(\xi \cos \xi - \sin \xi). \end{aligned}\right\} \tag{33}$$

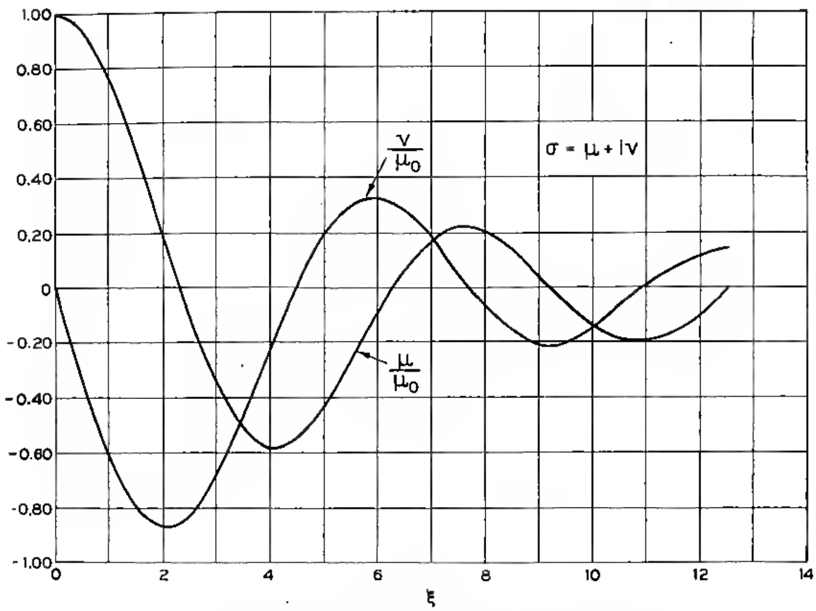

Fig. 5—Real and imaginary components of complex amplification factor of negative grid triodes versus transit angle.

The significance of (32) is at once apparent when it is compared with the classical form of the equation representing the alternating-current plate voltage, namely,

$$V_p = I_p r_0 - \mu V_g.$$

The plate resistance r_0 has now become complex as likewise has the amplification factor μ. Values of the plate impedance

$$z_p = r + ix$$

are the same as those obtained for the diode and are plotted in Figs. 1 and 3. Values of the amplification factor

$$\sigma = \mu + i\nu$$

are shown in Figs. 5 and 6.

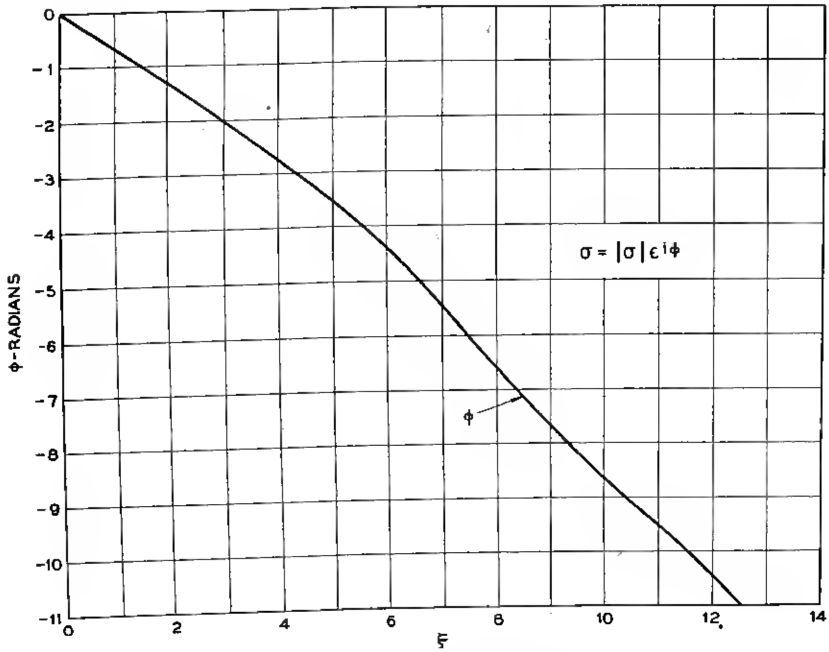

Fig. 6—Phase angle of amplification factor of negative grid triodes versus transit angle.

It is evident that radical changes in the phase angles existing between the grid voltage and plate current are present when the transit time becomes appreciable in comparison with the period of the applied electromotive force. The plate impedance decreases in magnitude as also does the magnitude of the amplification factor. However, the ratio of the two, namely, σ/z, maintains a fairly constant magnitude as shown in Fig. 7, whose phase angle nevertheless rotates continually in a negative direction becoming equal to 3 radians when ξ is 2π.

The interelectrode capacity between cathode and plate is included in the fundamental relations here employed. This inclusion exhibits one important difference between (32) and the classical case. At low frequencies, the equivalent circuit represented by (32) degenerates into that shown on Fig. 8. The capacity branch exists in parallel with the

resistive branch and they are both in series with the effective generator σe_g, whereas in the classical picture the capacity branch shunts the effective generator and plate resistance which are in series with each other. Practically the difference between the two equivalent circuits is negligible except at extremely high frequencies. The following physical viewpoint supports the newer picture.

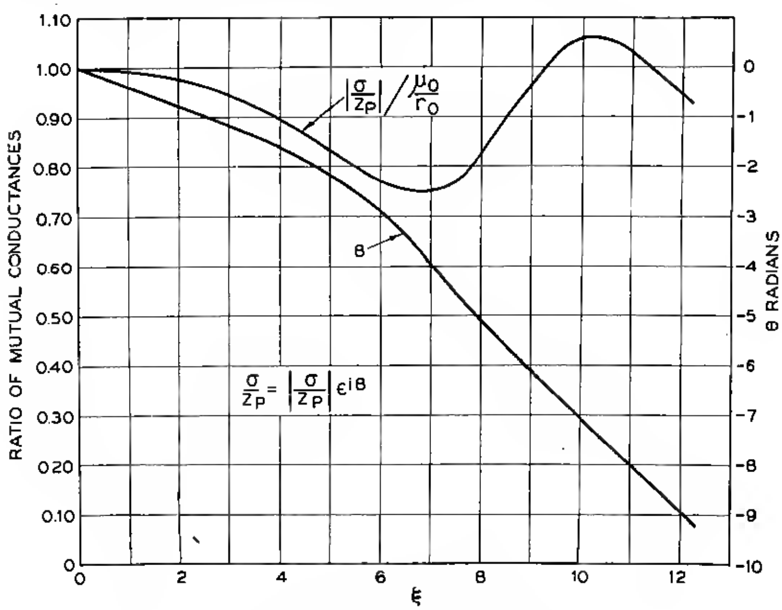

Fig. 7—Magnitude of complex mutual conductance of negative grid triodes versus transit angle.

As pointed out, the action of the grid is exerted mostly on the region of dense space charge existing very near the cathode and variations in the grid potential act on the velocities of the emerging electrons, thus producing the equivalent generator of the plate circuit. The plate current consists of conduction and displacement components whose sum is the same at all points in the cathode-plate path. Near the cathode, the conduction component comprises the whole current because of the high charge density and the effective generator acts in series with this current and hence in series with the path of the displacement current into which the character of the total current gradually changes as the plate is approached.

Strictly speaking, the equivalent circuit corresponding to (32) exists, not between the plate and cathode, but between the plate and the potential minimum near the cathode which is caused by the finite

velocities with which electrons are emitted from the cathode. Practically, the difference is negligible except at extremely high frequencies. Since the impedance between the cathode and potential minimum is small compared to the plate impedance, its effect is merely to add a loss to the system which increases with frequency since the plate impedance approaches a capacity as the frequency approaches infinity.

The grid cathode path presents less difficulty, although a somewhat less rigorous treatment is given here. As pointed out, the force from the grid acts on the high charge density region existing near the potential minimum. The impedance between cathode and grid, therefore, consists of two parts in series; namely, capacity between grid and potential minimum and impedance between potential minimum and cathode, the latter part of this impedance being common both to plate- and grid-current paths.

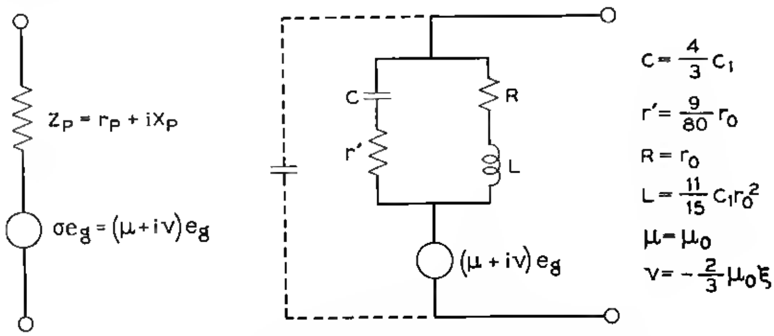

Fig. 8—Equivalent network of plate-cathode path of negative grid triodes for transit angles less than 0.3 radian.

If we were to connect the grid and cathode terminals of such a triode to a capacity bridge and measure the capacity existing there when the tube was cold and when the cathode was heated, we should find that the capacity would exhibit a slight increase in the latter case. The reason for this increase may best be explained by noting that in the cold condition the electrostatic force from the grid is exerted on the cathode itself, whereas in the heated state, the force acts on the electrons near the potential minimum, thus resulting in an increased capacity in series with a resistive component.

In some measurements of the losses in coils which were made at a frequency of 18 megacycles, J. G. Chaffee of the Bell Telephone Laboratories has found that a loss existed between grid and cathode of vacuum tubes which was much greater than can be accounted for by any of the dielectrics used and which was present only when the tube

filament was hot. This loss increased with frequency in the manner characteristic of that of the capacity-resistance combination between cathode and grid which was described above. Present indications are that, at least in part, the loss may be ascribed to the resistance existing between the cathode and the region of potential minimum.

Of the three current paths through the tube, one more still remains to be considered. This is the grid-plate path. The relations involved here are more readily seen by considering first a low-frequency example. Here the electron stream passes through the spaces between grid wires, afterward diverging as the plate is approached. Electrostatic force from the grid acts not only on the plate but also on the electrons in the space between. It is evident, then, that the path which, when the cathode was cold, constituted a pure capacity changes into an effective capacity different from the original in combination with a resistive component. The losses would be expected to increase with frequency just as they did in the grid-cathode type. The change in grid-plate impedance is particularly noticeable when it is attempted to adjust balanced or neutralized amplifier circuits with the filament cold, in which case the balance is disturbed when the cathode is heated.

As yet, no accurate expression for this grid-plate impedance has been obtained, either at the low frequencies where transit times are negligible or at the higher frequencies now particularly under investigation. The reason for this lies in the repelling force on the electron stream of the negative grid so that the assumption of current flow in straight parallel lines is not valid in so far as current from the grid to the plate is involved.

It has been shown that both the cathode-grid path and the grid-plate path contain resistive components with corresponding losses which increase with increase of frequency. This loss may be cited as a reason why triodes with negative grids cease to oscillate at the higher frequencies. If it were not for these losses, external circuits could be attached to the tube having such phase relations as to satisfy oscillation conditions, so that the negative grid triode could be utilized in the range which is now covered by the triode with positive grid.

V. Triodes with Positive Grid and Slightly Positive Plate

When the grid of a three-element tube is operated at a high positive potential with respect both to cathode and plate, electrons are attracted toward the grid, and the majority of them are captured on their first transit. Those which pass through the mesh and journey toward the plate will be captured by the plate if its potential is sufficiently positive with respect to the cathode.

In general, space-charge conditions existing between grid and plate are quite complicated. An analysis has been made by Tonks [4] which indicates several distinct classes of space-charge distribution which are possible. In the first place so few electrons may pass the grid mesh that no appreciable space charge is set up between there and the plate. In this instance a positive plate will trap them all, whereas a negative plate will return them all toward the grid. Second, with a fixed positive plate potential an increase in the number of electrons which pass the grid mesh will result in a depression of the potential distribution as illustrated at (*a*) by the curves in Fig. 9. This depression will con-

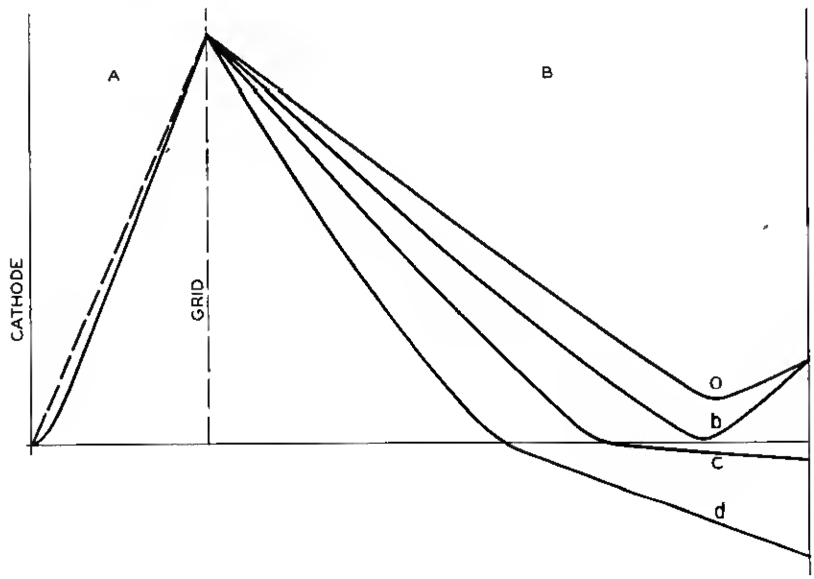

Fig. 9—Potential distributions in positive grid triodes.

tinue to increase until a potential minimum is formed. When this potential minimum becomes nearly the same as that of the cathode, either of several things may occur. If the minimum is just above the cathode potential, all electrons will pass that point and eventually reach the plate. However, an extremely small increase in the number of electrons will cause the potential minimum to become equal to the cathode potential. When this happens some of the electrons will be turned back and travel again toward the grid. These will increase the charge density existing and, therefore, cause a further depression in the potential resulting in a mathematical discontinuity so that the

[4] L. Tonks, "Space Charge as a Cause of Negative Resistance in a Triode and Its Bearing on Short-Wave Generation," *Phys. Rev.*, Vol. 30, p. 501; October (1927).

curve of the potential suddenly changes its shape with a resulting change in plate current. Again, the plate may be operated at a negative potential. In this case, none of the electrons will reach it and the potential distribution curves have the character illustrated at (*c*) and (*d*) in Fig. 9.

In attempting to apply the fundamental relations to this grid-plate region, we must choose our origin at a point where the potential distribution curve touches the zero axis and is tangent to it. Whenever such a point exists, the relations may be applied as described below. Even when this condition does not exist inside the vacuum tube, there may be a virtual cathode existing outside of the plate.

Whenever all of the electrons passing the grid reach the plate the general equations may be applied in a straightforward manner with the origin taken at the virtual cathode. Whenever some of the electrons are turned back toward the grid, slightly different equations are required, although they may be applied in the same manner. These modified equations will be derived and discussed after the application of the equations already derived has been made to the case where all of the electrons reach the plate.

Choosing the origin for this latter case at the point of zero potential or virtual cathode, we can compute the impedance between the grid-plane and the virtual cathode when we know the alternating-current velocities with which the electrons pass through the grid-plane. This has been found for the condition of complete space charge between cathode and grid and was given by (25). Likewise, it can be found on the supposition that no space charge exists in the cathode-grid region and the result will be calculated later. Thus, two limiting cases are available for numerical application.

In order to prevent confusion for the grid—virtual-cathode region where the electron flow is toward the origin rather than away from it, as was assumed in the derivation of the fundamental relations, it will be convenient to change the symbol for transit angle from ξ to $-\zeta$. This will automatically take care of all algebraic signs, currents and velocities now being considered positive when directed towards the origin.

Since we are computing the impedance between an origin at the virtual cathode and the grid plane we may apply (24) to find the potential difference, getting

$$V_1 - V_1' = -\left(\frac{2m}{e}\frac{\alpha^3}{9p^2}\right)(M + iN)[(1 - \cos\zeta) - i(\zeta - \sin\zeta)]$$
$$- i\left(\frac{2m}{e}\frac{\alpha^3\beta}{9p^4}\right)\left(\frac{1}{6}\zeta^3 - \frac{4}{\zeta}(1 - \cos\zeta) + 2\sin\zeta\right), \quad (34)$$

where V_1' is the potential at the virtual cathode.

This relation is of the form

$$V_g - V_p = -(M+iN)\left(\frac{2m}{e}\frac{\alpha^3}{9p^2}\right)[(1-\cos\zeta)-i(\zeta-\sin\zeta)]+J_pZ_p, \quad (35)$$

where J_p is the plate current, and Z_p is the effective impedance:

$$Z_p = -i\left(\frac{2m}{e}\frac{\alpha^3\beta}{9p^4A}\right)\left(\frac{1}{6}\zeta^3 - \frac{4}{\zeta}(1-\cos\zeta)+2\sin\zeta\right). \quad (36)$$

In terms of the cold capacity C_1 between plate and grid plane this becomes

$$Z_p = -\frac{i}{pC_1}\frac{6}{\zeta^3}\left[\frac{1}{6}\zeta^3 - \frac{4}{\zeta}(1-\cos\zeta)+2\sin\zeta\right],$$

which is plotted in Fig. 10.

The form of (35) shows that the equivalent network between the plane of the grid and the plate may be represented by an equivalent generator acting in series with the impedance, Z_p. This is evidenced by the fact that the velocity $M + iN$ with which the electrons pass the grid, may be expressed in terms of the grid potential V_g by means of conditions between the grid and cathode. When complete space charge exists near the cathode, these conditions are expressed by (25) and (26). On the other hand, tubes with positive grid are sometimes operated with inappreciable space charge between grid and cathode. In this event, a similar analysis leads to values for the alternating-current velocity and potential at the grid as follows:

$$U_1 = M + iN = -\frac{\beta}{p^2}\left[\left(\frac{\eta-\sin\eta}{\eta}\right)+i\left(\frac{1-\cos\eta}{\eta}\right)\right], \quad (37)$$

$$V_g = i\frac{4\pi x}{p}A = \frac{iA}{pC}, \quad (38)$$

where η is the transit angle in the absence of space charge, and C is the electrostatic capacity between unit area of cathode and of grid plane. The right-hand side of (38) does not contain a minus sign because of the assumed current direction which is away from the cathode, as is also the convention employed in (25) and (26) where the electron charge e is a positive number.

The relations given by (35) allow the potential difference between grid and plate to be determined in terms of the total current flowing to the plate, and the total current flowing from the cathode, which ap-

pears in the velocity factor $M + iN$. In the usual case some of the alternating current flows to the grid wires and is returned through an external circuit connected to the grid. If the impedance between grid and plate is desired it is necessary to find the relation which this grid current bears to the total cathode and plate currents and to the alternating-current potentials. The calculations involved are extremely complicated because the assumption of current flow in straight lines between parallel planes is far from representing the actual conditions

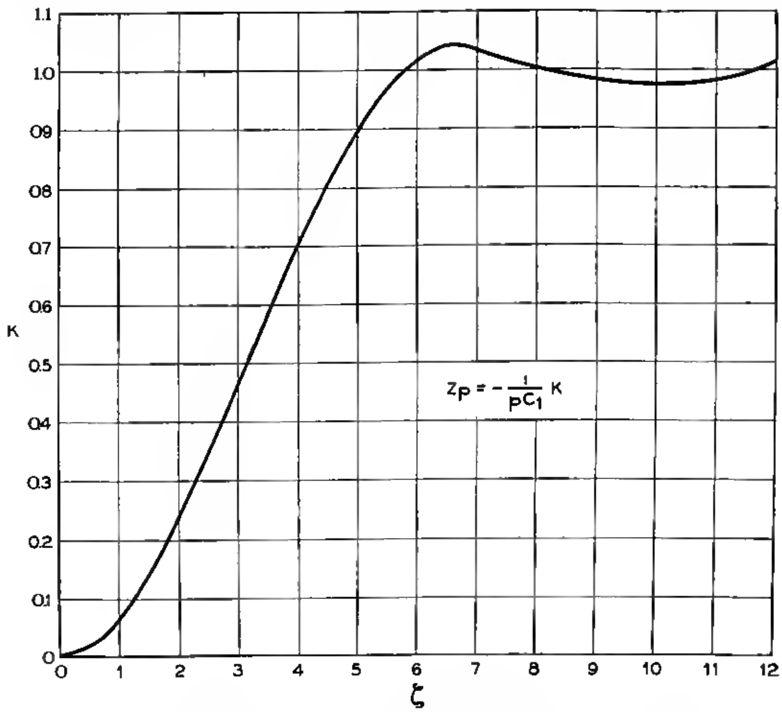

Fig. 10—Plate impedance of positive grid triodes with slightly positive plate.

in the immediate neighborhood of the grid wires. Rather than attempting an analysis of these conditions at the present time, we shall content ourselves with results already obtained, since they are applicable to the special case, which can be realized approximately in experiment, where the grid is connected to a radio-frequency choke coil of sufficiently good characteristics to prevent it from carrying away any alternating current. For this special case the current J_1 is the same both in the cathode region and in the plate region, and all encumbering assumptions involving different paths for the conduction and displace-

ment components of the current in the neighborhood of the grid wires have been done away with.

The application of the equations to this special case is dealt with in the section of this paper devoted to positive grid oscillators. Before these oscillators can be treated comprehensively, a further extension of fundamental theory is necessary. This extension comes about because positive grid oscillators are often operated with a slightly negative potential applied to the plate.

VI. Triodes with Positive Grid and Negative Plate[5]

When consideration is directed to tubes operating with positive grid but negative plate, the fundamental underlying theory must again be investigated. The reason for this lies in the fact that all electrons which penetrate through the meshes of the grid are turned back before they reach the plate, so that in the grid-plate space there are two streams of electrons moving in opposite directions. The effect of this double value for the velocity may readily be calculated in so far as direct-current components, only, are concerned. We have merely to note that the charge density is double the value which it would have in the presence of those electrons which are moving in one direction, only, so that the correct relations are obtained from the equations already derived by taking twice the value of direct current in one direction.

When alternating-current components are considered, however, matters are more complicated, but not difficult. To see what the actual relations are, let there be two possible values at any point for the instantaneous velocity, and call these two values U_a and U_b, respectively. Then the relation between force and acceleration becomes

$$\frac{eE}{m} = \frac{dU_a}{dt} = \frac{dU_b}{dt}. \tag{39}$$

Hence, at a given value of x we have by integration

$$U_a = U_b + \text{constant}.$$

But, when both values of velocity are separated into their components according to (5) we have from (39)

$$U_{a0} + U_{a1} + \cdots = U_{b0} + U_{b1} + \cdots + \text{constant}.$$

[5] Since the publication of this paper in the *Proceedings* of the *Institute of Radio Engineers*, several questions have been raised regarding the treatment presented in sections VI and VII. These are being investigated and will form the basis of another paper.

By equating corresponding terms, we find

$$\left.\begin{aligned} U_{a0} &= U_{b0} + \text{constant}, \\ U_{a1} &= U_{b1}, \\ U_{a2} &= U_{b2}, \text{ etc.} \end{aligned}\right\} \tag{40}$$

The first of these equations is trivial when the boundary conditions are inserted, for then it appears that $U_{a0} = - U_{b0}$ and the equation merely states that at a given value of x the direct-current velocity component is not a function of time.

The second equation is much more enlightening and tells us that although two values of the direct-current velocity may be present, nevertheless there is only a single value for the alternating-current component. The same conclusion holds for the higher order velocity components. This conclusion supplies the key for the solution of the general equations when applied to the stream of electrons moving in both directions between the grid and plate of the tube.

In general, the total current may be written

$$J = P_a U_a + P_b U_b + \frac{1}{4\pi} \frac{\partial E}{\partial t}. \tag{41}$$

If Σ is the total area of each of the electrode planes and

$$\Sigma = a + b,$$

where a and b are constants to be defined later, (41) may be written as follows:

$$J\Sigma = \left(P_a U_a \frac{\Sigma}{a} + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right) a + \left(P_b U_b \frac{\Sigma}{b} + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right) b. \tag{42}$$

In this expression, the two streams of current are clearly separated if a and b are taken so that [6]

$$P_a \frac{\Sigma}{a} = P \qquad \text{and} \qquad P_b \frac{\Sigma}{b} = P, \tag{43}$$

where P is the total charge density, equal to the sum of P_a and P_b.

The total current may now be expressed in terms of velocities, only, giving similarly to the transition from (1) to (3),

$$4\pi \frac{e}{m} J\Sigma = a \left(U_a \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right)^2 U_a + b \left(U_b \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right)^2 U_b. \tag{44}$$

[6] A more rigorous analysis, involving mean values of the motions of individual electrons, leads to the same result.

When U_a and U_b are each separated into their components according to (5), so that (44) may be resolved into a system of equations, we have for the first two equations, analogous to (6) and (7),

$$4\pi \frac{e}{m} J_0 \Sigma = (a - b) \left[U_0 \frac{\partial}{\partial x} \left(U_0 \frac{\partial U_0}{\partial x} \right) \right] \tag{45}$$

and

$$\beta\Sigma = (a+b) \left[U_0 \frac{\partial}{\partial x} \left(U_0 \frac{\partial U_1}{\partial x} + U_1 \frac{\partial U_0}{\partial x} \right) + \frac{\partial^2 U_1}{\partial t^2} + U_1 \frac{\partial}{\partial x} \left(U_0 \frac{\partial U_0}{\partial x} \right) \right]$$
$$+ (a-b) \left[U_0 \frac{\partial}{\partial x} \left(\frac{\partial U_1}{\partial t} \right) + \frac{\partial}{\partial t} \left(U_0 \frac{\partial U_1}{\partial x} + U_1 \frac{\partial U_0}{\partial x} \right) \right], \tag{46}$$

where the components of U_b have been expressed in terms of those of U_a by means of (40) and the relation that $U_{b0} = - U_{a0}$.

The solution of (45) is, as before,

$$U_0 = \alpha x^{2/3}. \tag{47}$$

where

$$\alpha = \left(18\pi \frac{e}{m} J_{0a} \frac{\Sigma}{a} \right)^{1/3}.$$

Before attempting to solve (46) we make a change of variable as in (10), writing

$$\xi = \frac{3p}{\alpha} x^{1/3} \qquad \text{and} \qquad U_1 = \frac{\omega}{\xi}.$$

This gives from (46)

$$\beta\Sigma = \left[\frac{p^2}{\xi} \frac{\partial^2 \omega}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial^2 \omega}{\partial t^2} \right] \Sigma + (a - b) \left(\frac{2p}{\xi} \frac{\partial^2 \omega}{\partial \xi \partial t} \right). \tag{48}$$

In finding a solution for this, we shall restrict ourselves to the case where all of the electrons turn back at the virtual cathode, so that $a = b$ and therefore the last term of (48) vanishes. The solution of the remaining equation is then,

$$U_1 = - \frac{\beta}{p^2} \left[\sin pt + \frac{1}{\xi} F_1(i\xi + pt) + \frac{1}{\xi} F_2(i\xi - pt) \right], \tag{49}$$

which is analogous to (12).

Again, assuming the two arbitrary functions to have the form,

$$\left. \begin{aligned} F_1(i\xi + pt) &= a \sin (i\xi + pt) + b \cos (i\xi + pt), \\ F_2(i\xi - pt) &= c \sin (i\xi - pt) + d \cos (i\xi - pt), \end{aligned} \right\} \tag{50}$$

and inserting the boundary conditions, (14) and (15), we have, in complex form,

$$U_1 = (M + iN) \frac{\xi_1 \sinh \xi}{\xi \sinh \xi_1} - \frac{\beta}{p^2}\left(1 - \frac{\xi_1 \sinh \xi}{\xi \sinh \xi_1}\right), \tag{51}$$

which is a simpler equation than its analogue (18). The potential is obtained as in (24) giving,

$$\begin{aligned} V_1 = &- \left(\frac{\alpha^3 m}{9p^2 e}\right)(M + iN)\frac{\xi_1}{\sinh \xi_1}[\xi \sinh \xi + i(\xi \cosh \xi - \sinh \xi)] \\ &+ \frac{\alpha^3 m\beta}{9p^4 e}\left[\left(\xi^2 - \frac{\xi_1 \xi \sinh \xi}{\sinh \xi_1}\right)\right. \\ &\left. + i\left(\frac{\xi^3}{3} - \frac{\xi_1}{\sinh \xi_1}(\xi \cosh \xi - \sinh \xi)\right)\right] + \text{constant}. \end{aligned} \tag{52}$$

The alternating-current potential difference between the grid and the virtual cathode where all of the electrons are turned back may be obtained immediately from (52). As before, the variable ζ will be substituted for ξ to show that the grid-plate region is considered, and currents and velocities will be considered positive when directed towards the origin at the virtual cathode. Thus, from (52)

$$\begin{aligned} V_g - V_p = &- \frac{\alpha^3 m}{9p^2 e}(M + iN)[\zeta^2 + i(\zeta^2 \coth \zeta - \zeta)] \\ &+ \frac{\alpha^3 m\beta}{9p^4 e} i\left[\frac{\zeta^3}{3} - \zeta^2 \coth \zeta + \zeta\right]. \end{aligned} \tag{53}$$

The velocity, $(M + iN)$ may be expressed in terms of the alternating-current grid potential, V_g, so that the path between grid plane and virtual cathode may be represented by an effective generator in series with an impedance, as was done in (34), (35), and (36).

VII. Oscillation Properties of Positive Grid Triodes [5]

The oscillation properties of the positive grid triode are next to be investigated. In the usual experimental procedure, an external high-frequency circuit is connected between the grid and the plate of the tube. It is unfortunate that this particular arrangement greatly complicates the theoretical relations. Accordingly, a slightly modified experimental set-up will be considered. This modification consists in connecting the external circuit between the cathode and plate of the tube, rather than between grid and plate. Experimental tests have shown that the modified circuit exhibits the same general phenomena

[5] Loc. cit.

as the more usual one, the difference being mainly one of mechanical convenience in securing low-loss leads between the tube and the external circuit.

The modified circuit, then, will be employed for analysis, and the assumption will be made that the necessary direct-current connections are made through chokes which are sufficiently good so that it may be considered that no external high-frequency impedance is connected between either the grid and the plate, or between the cathode and the grid.

It is easy to see that under these conditions there can be no high-frequency current carried away by the grid. It follows that for plane-parallel structures, the alternating-current density, J_1, will be the same both in the cathode-grid region and in the grid-plate region. The arrangement thus reduces the problem to the consideration of the single current, J_1, and the resulting potential difference between cathode and plate.

There are several possible combinations of direct-current biasing potentials. For the first of these, the plate will be supposed to be biased at a potential sufficiently positive to collect all electrons which are not captured by the grid on their first transit. Complete space charge will be assumed both in the cathode region and in the plate region.

Under these conditions, we have the grid-cathode potential difference given by (26) and the grid-plate potential difference given by (35), where the velocity, $M + iN$, is given by (25). We can write,

$$\begin{aligned} V_p - V_c &= (V_p - V_g) + (V_g - V_c) \\ &= -\,[\text{Eq. } 35] + [\text{Eq. } 26]. \end{aligned} \tag{55}$$

It will be remembered that the current was assumed to be positive in (26) when directed away from the origin, and positive in (35) when directed toward the origin. Therefore, since the same current exists in both regions, and they are joined together at the grid, the sign of the current J_1 remains the same in both (35) and (26), its direction being from cathode to plate. The impedance looking into the cathode-plate terminals may be obtained from (55) by dividing by the amplitude A of J_1 and reversing the sign of the result to correspond to a current from plate to cathode. Letting

$$Z_0 = R_0 + iX_0 \tag{56}$$

represent the impedance looking into the cathode-plate terminals, we can write the result as follows

$$\begin{aligned} R_0 = -\frac{12r_0}{\zeta^4}\Bigg[&\left(1 + \cos\eta - \frac{2}{\eta}\sin\eta\right)(1 - \cos\zeta) \\ &- \left(\frac{2}{\eta} - \sin\eta - \frac{2}{\eta}\cos\eta\right)(\zeta - \sin\zeta) \\ &+ (2\cos\eta\sin\eta - 2)\Bigg], \end{aligned} \tag{57}$$

$$\begin{aligned} X_0 = -\frac{12r_0}{\zeta^4}\Bigg\{ &\left(1 + \cos\eta - \frac{2}{\eta}\sin\eta\right)(\zeta - \sin\zeta) \\ &+ \left(\frac{2}{\eta} - \sin\eta - \frac{2}{\eta}\cos\eta\right)(1 - \cos\zeta) \\ &+ \left[\frac{1}{6}\zeta^3 - \frac{4}{\zeta}(1 - \cos\zeta) + 2\sin\zeta\right] \\ &+ \left[\eta + \frac{1}{6}\eta^3 - 2\sin\eta + \eta\cos\eta\right]\Bigg\}, \end{aligned} \tag{58}$$

where η is the transit angle from cathode to grid, ζ is the transit angle from grid to virtual cathode at the plate, and r_0 is the zero-frequency resistance which would be present in a diode having the grid-plate dimensions, and the same operating direct-current voltages and current densities which occur in the grid-plate region of the triode under consideration.

Fig. 11 shows graphically the relation between R_0 and X_0 for a wide frequency range, in terms of the reference resistance, r_0. Curve A is drawn for the hypothetical condition that $\eta = \zeta$, so that the tube is exactly symmetrical about the grid. Actually such a condition could not be attained, since the grid captures some of the electrons, leaving fewer for producing space charge near the plate. The grid-plate dimension would accordingly have to be increased in order to secure the space charge, but this would cause the transit angle ζ to become larger than η. However, despite the fact that it does not correspond to a physically realizable condition, curve A is nevertheless of use in indicating the limit which is approached as the grid capture fraction is made smaller and smaller.

Curves B and C correspond to values of grid-plate transit angle equal respectively to two and three times the cathode-grid transit angle. Both these curves represent conditions which may readily be obtained experimentally, and indeed, curves lying much closer to A than does the curve B may be secured. For example, the general relation for the ratio of the transit angles in terms of the direct currents J_a and J_b in the cathode and in the plate region, respectively, when

complete space charge exists in both regions, is,

$$\frac{\zeta}{\eta} = \sqrt{\frac{J_a}{J_b}}.$$

Suppose that the grid captured half of the electrons. Then the ratio of transit angles would be 1.41. This would result in a curve lying between A and B in Fig. 11.

The numbers, $\pi/2$, π, and so forth, which are attached to the curves in Fig. 11 show the values of the grid-plate transit angle, ζ, which correspond to the points indicated.

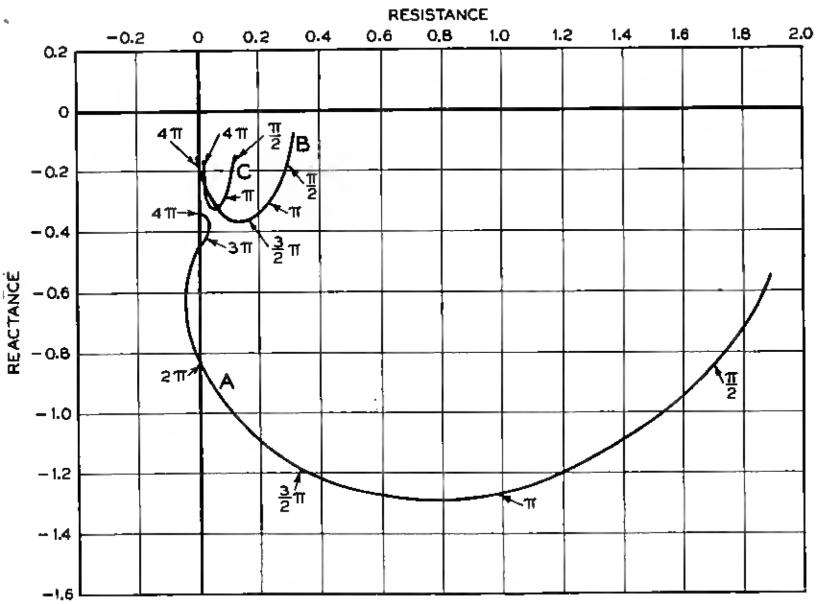

Fig. 11—$R_0 - X_0$ diagram for positive grid, slightly positive plate triode with cathode space charge.

Curve A, $\eta = \zeta$
Curve B, $\eta = 1/2\zeta$
Curve C, $\eta = 1/3\zeta$

We now come to the problem of obtaining information about the oscillation properties of a tube from a set of curves such as those shown in Fig. 11. In a very loose way, and without proof we may state the results of an extension of Nyquist's[7] rule as follows:

If an $R - X$ diagram, which in general may include negative as well as positive frequencies, encircles the origin in a clockwise direc-

[7] H. Nyquist, "Regeneration Theory," *Bell Sys. Tech. Jour.*, Vol. 11, p. 126; January (1932).

tion, then the system represented by the diagram will oscillate when the terminals between which the impedance was measured are connected together.

Verification of this rule, together with further extension to more general cases are expected to be discussed in a subsequent paper. For the present, its validity will have to be accepted on faith, but with the assurance that the applications employed in this discussion are readily capable of demonstration.

Returning to consideration of the positive grid triode with complete space charge on both sides of the grid, and a slightly positive plate, whose $R - X$ diagram is given in Fig. 11, we see at once that the diagram does not encircle the origin as it stands. Of course only positive values of frequency are included in the curves as they are shown. The inclusion of negative frequencies (never mind their physical meaning) would produce a curve which would be the image of the curve shown, a reflecting mirror being regarded as a plane perpendicular to the paper, and containing the R-axis. The curve A, for instance, would have its part corresponding to negative frequencies lying above the R-axis and forming an image of the part lying below. This is shown by the dotted curve in Fig. 12.

It is obvious that the curve of Fig. 12 will encircle the origin or not depending on what happens at infinite frequencies. However, the slightest amount of resistance in the leads to the tube will be sufficient to move the curve to the right and thus exclude the origin. This means that no oscillations would be obtained if an alternating-current short were placed between plate and cathode. The result, although in accord with experiment, is not particularly useful. The important thing is to find whether the curve can be modified by the addition of a simple electrical circuit in such a way that the origin of the resulting $R - X$ diagram for the combination of tube and circuit is encircled in a clockwise direction.

Suppose that a simple inductance is connected in series with the plate lead, and the impedance diagram of the series combination of tube and inductance is plotted. For this arrangement, the $R - X$ diagram of Fig. 12 would be modified as shown in Fig. 13. Here the part of the curve corresponding to negative values of resistance has been pushed upward until the origin is enclosed within a loop which encircles it in a clockwise direction. It is therefore to be expected that oscillations will result. As to their frequency, we can say that the grid-plate transit angle must be at least as great as 2π for this particular example. This follows by supposing a certain amount of resistance to be added in series with the circuit. The effect of this resistance will

be to move the curves on Fig. 13 bodily to the right. The lowest frequency which will just allow the origin to be included within the loop when the series resistance is reduced to zero and the inductance is adjusted, corresponds to a grid-plate transit angle of 2π.

It must be remembered that the foregoing details apply only to curve A of Fig. 11, and it has already been pointed out that curve A

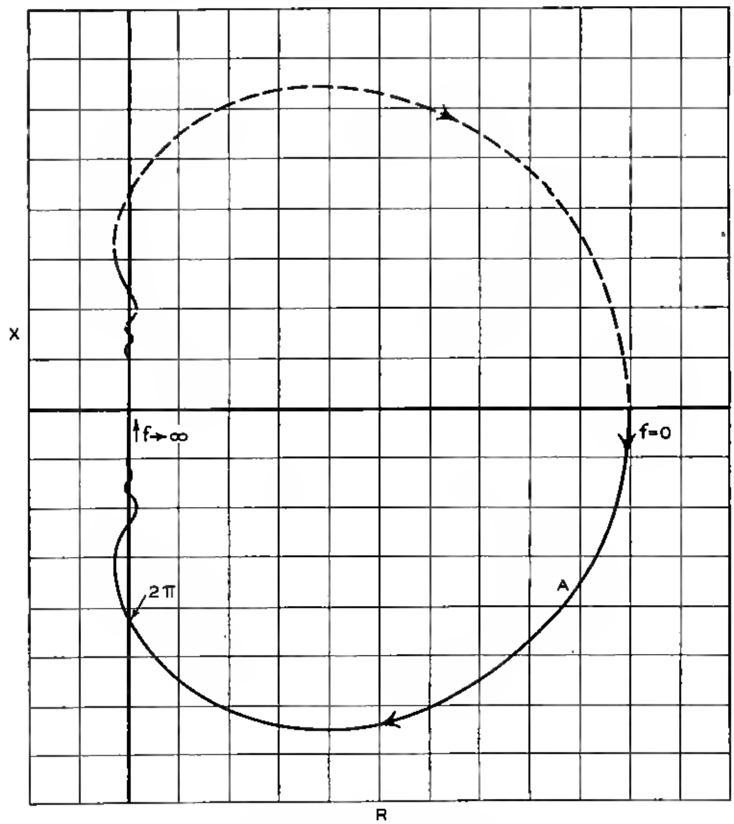

Fig. 12—Curve A of Fig. 11 together with the image corresponding to negative frequencies.

represents a limit which can be approached in practice, only as the grid capture fraction is made smaller and smaller. Curve B can well be duplicated in experiment. For this case, the lowest frequency at which oscillations may be expected is much higher than before, since the transit angle must be equal to 4π before the resistance becomes negative. Actually, conditions intermediate between the two curves may be realized, so that from a practical standpoint the transit angle

must be in the neighborhood of 3π before we may expect to secure oscillations.

This would correspond to a frequency somewhat higher than is often associated with this type of oscillation. It must be remembered however, that the particular case considered was that of a tube with its plate at a slightly positive potential, whereas the majority of the experimental frequency observations were made with the plate either slightly negative, or, if positive, adjusted so that a virtual cathode was formed inside the tube, and many of the electrons were turned back

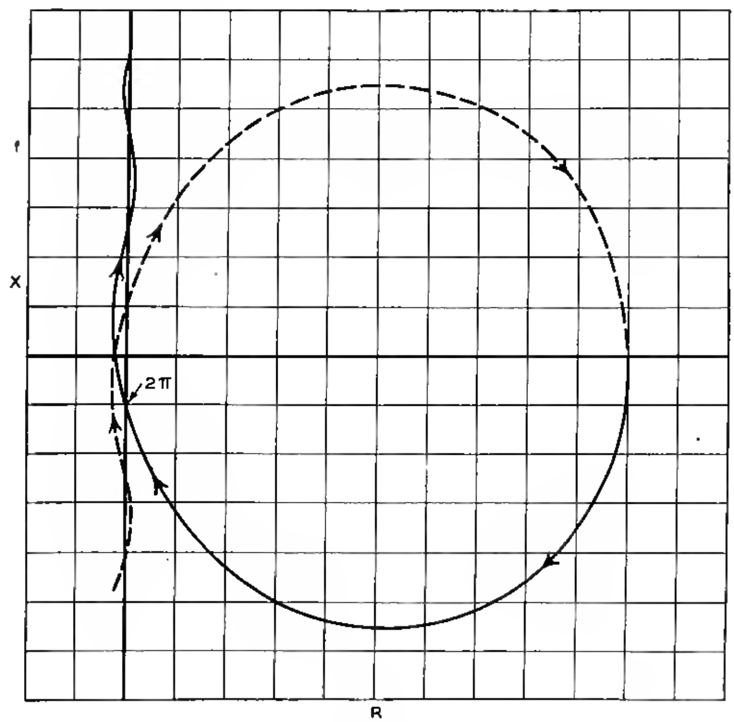

Fig. 13—Modification of Fig. 12 produced by added inductance.

before they reached the plate. The curves of Fig. 11 do not apply to these cases.

Therefore, let us see what happens when the plate is operated at a negative potential so that all of the electrons are turned back before they reach it. At the outset, it should be remarked that this condition does not prohibit the presence of direct-current plate current *after the oscillations have built up to a finite amplitude.* The analysis applies to the requirements for the starting of the oscillations, only, so that

if the plate fluctuates in potential by a very small amount, as it does for incipient oscillations, and hence does not become positive during the alternating-current alternation, then no direct-current plate current can occur when the plate is biased negatively. After oscillations have built up to an appreciable amplitude, the presence of plate current is not only possible, but is in fact to be expected.

We have at hand the mathematical tools with which to compute our $R - X$ diagram for the negative plate triode with complete space charge near the cathode. Thus, instead of substituting (35) in (55) we must substitute (53). Since complete space charge is still postulated near the cathode, (26) and (25) are still applicable. The result is:

$$\begin{aligned} R_0 = -\frac{12r_0}{\zeta^4}\Bigg[& \left(1 + \cos\eta - \frac{2}{\eta}\sin\eta\right)\zeta^2 \\ & - \left(\frac{2}{\eta} - \sin\eta - \frac{2}{\eta}\cos\eta\right)(\zeta^2\coth\zeta - \zeta) \\ & + (2\cos\eta + \eta\sin\eta - 2)\Bigg], \end{aligned} \tag{59}$$

$$\begin{aligned} X_0 = -\frac{12r_0}{\zeta^4}\Bigg[& \left(\frac{2}{\eta} - \sin\eta - \frac{2}{\eta}\cos\eta\right)\zeta^2 \\ & + \left(1 + \cos\eta - \frac{2}{\eta}\sin\eta\right)(\zeta^2\coth\zeta - \zeta) \\ & + (\tfrac{1}{3}\zeta^3 - \zeta^2\coth\zeta + \zeta) + (\eta + \tfrac{1}{6}\eta^3 - 2\sin\eta + \eta\cos\eta)\Bigg], \end{aligned} \tag{60}$$

and the corresponding diagram is shown in Fig. 14. Here the curve A shows oscillation possibilities for transit angles as small as $3/2\pi$, while a much greater amount of resistance would have to be added to the circuit in order to eliminate the negative resistance and so stop the oscillations. In all, then, this method appears to be a better way of operating the system than with the positive plate, and this conclusion is substantiated by experimental observations.

As before, an increase in the grid capture fraction moves the oscillation region up to higher frequencies.

In both of the examples cited above, and represented by Figs. 11 and 14, respectively, complete space charge was assumed near the cathode. The effect of decreasing the cathode heating current so that this charge becomes negligible may be computed by employing (37) in place of (25), and (38) in place of (26).

The resulting equations for a slightly positive plate are,

$$R_0 = -\frac{12r_0}{\zeta^4}\left[\left(\frac{\eta-\sin\eta}{\eta}\right)(1-\cos\zeta)-\left(\frac{1-\cos\eta}{\eta}\right)(\zeta-\sin\zeta)\right], \tag{61}$$

$$X_0 = -\frac{12r_0}{\zeta^4}\left\{\left(\frac{\eta-\sin\eta}{\eta}\right)(\zeta-\sin\zeta)+\left(\frac{1-\cos\eta}{\eta}\right)(1-\cos\zeta)\right.$$

$$\left.+\left[\tfrac{1}{6}\zeta^3-\frac{4}{\zeta}(1-\cos\zeta)+2\sin\zeta\right]+\tfrac{1}{4}\eta\zeta^2\right\}. \tag{62}$$

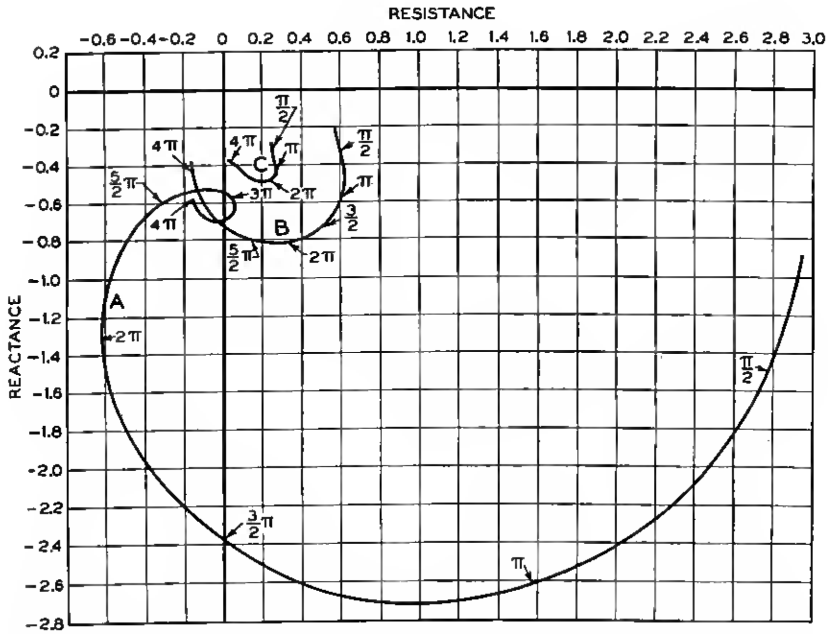

Fig. 14—$R_0 - X_0$ diagram for positive grid, slightly negative plate triode with cathode space charge.

Curve A, $\eta = \zeta$
Curve B, $\eta = 1/2\zeta$
Curve C, $\eta = 1/3\zeta$

The corresponding $R - X$ diagram is given in Fig. 15.

Again, the equations for a negative plate and no cathode space charge are,

$$R_0 = -\frac{12r_0}{\zeta^4}\left[\left(\frac{\eta-\sin\eta}{\eta}\right)\zeta^2-\left(\frac{1-\cos\eta}{\eta}\right)(\zeta^2\coth\zeta-\zeta)\right], \tag{63}$$

$$X_0 = -\frac{12r_0}{\zeta^4}\left[\left(\frac{1-\cos\eta}{\eta}\right)\zeta^2+\left(\frac{\eta-\sin\eta}{\eta}\right)(\zeta^2\coth\zeta-\zeta)\right.$$

$$+\left(\tfrac{1}{3}\zeta^3 - \zeta^2 \coth \zeta + \zeta\right) + \tfrac{1}{4}\eta\zeta^2 \Big], \tag{64}$$

and the $R - X$ diagram is shown in Fig. 16.

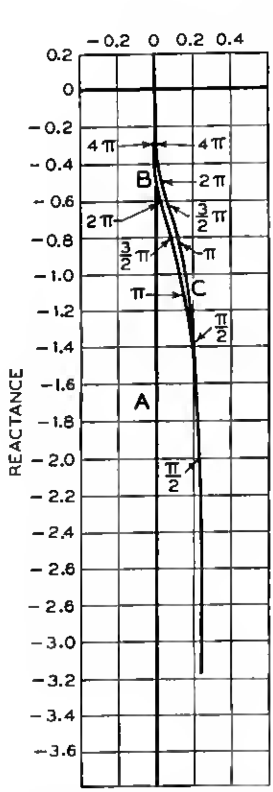

Fig. 15—$R_0 - X_0$ diagram for positive grid, slightly positive plate triode, without cathode space charge.

Curve A, $\eta = \zeta$
Curve B, $\eta = 1/2\zeta$
Curve C, $\eta = 1/3\zeta$

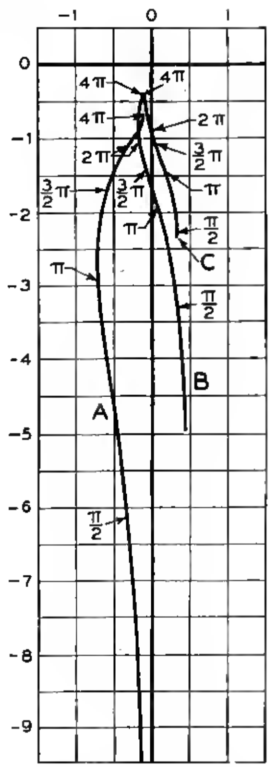

Fig. 16—$R_0 - X_0$ diagram for positive grid, slightly negative plate triode, without cathode space charge.

Curve A, $\eta = \zeta$
Curve B, $\eta = 1/2\zeta$
Curve C, $\eta = 1/3\zeta$

Inspection of Figs. 15 and 16 shows that the negative plate condition is greatly to be preferred when there is no cathode space charge. In fact, when account is taken of the difference in the scales for which Fig. 16 and the other three figures, 11, 14, and 15, are plotted, it is evident that the negative plate without space charge offers the greatest latitude in the adjustment of circuit condition. As in all of the cases, except Fig. 15, a small grid capture fraction is to be desired. If

curve A in Fig. 16 could be attained practically, it would be possible to secure oscillations even at low frequencies by connecting an inductance between the plate and cathode terminals. Curve B shows a low-frequency limit of a little less than $\frac{3}{2}\pi$ for the grid-plate transit angle.

One of the more important observations to be drawn from the curves of Figs. 11, 14, 15, and 16 is that if the inductance between plate and cathode is obtained by means of a tuned antiresonant circuit, then the circuit must be tuned to a frequency somewhat higher than the oscillation frequency. This is in order that it may effectively present an inductive impedance to the oscillating tube, so that the extended curves in the figures may encircle the origin in a clockwise direction.

Another conclusion is that there are so many different permutations and combinations of the operating conditions that it is small wonder that there have been a great many different "theories" and empirical frequency formulas advocated. For instance, operation under conditions giving an $R - X$ diagram which shows negative resistance over a small frequency range, only, such as A in Fig. 11, or B in Fig. 15, would give oscillations whose frequency would be much more nearly independent of the tuning of the external circuit than would conditions which resulted in a negative resistance over a wide frequency range, as at A in Fig. 16. In this latter case the external circuit exerts a large influence upon the frequency.

The data from which Figs. 11 to 16 were plotted are given in the appended tables. The final step in the calculation of these data was a multiplication by 12 which was performed on a slide rule. For all previous steps seven-place tables were employed because of the frequent occurrence of differences of numbers of comparable magnitude.

The effect on the frequency of a change in the operating voltages can be deduced inferentially from the curves. Thus, in general, the formulas for the transit angle have the form,

$$\xi = \frac{Kx}{\lambda\sqrt{V_0}}, \tag{65}$$

where x is the grid-origin distance,
λ is the wave-length,
V_0 is the grid potential,
K is a constant which depends on the mode of operation.

When the plate potential is changed by a relatively large amount the operation undergoes a transition from a limiting mode illustrated by one of the figures to another limiting mode shown on some other one of the figures.

On the other hand, a change in grid potential will act to change the

transit angles on the two sides of the grid in the same proportion. A modification of this generality occurs because the value of x in (65) will shift as the effective position of the virtual cathode moves about. Also, the complete space-charge condition near the plate becomes modified, and so the general relations become extremely variable. The partial space charge that exists with very negative values of plate potential, or with very high values of grid potential does not lend itself readily to mathematical treatment, so that intermediate conditions between complete and negligible space charge can be treated only by inference as to what happens between the two limiting conditions.

With inappreciable space charge on both the plate and the cathode sides of the grid, there can be no oscillations at all, since all impedances then approach pure capacities, with no negative resistance components.

A word concerning the so-called "dwarf" waves is in order before this general theoretical discussion is completed. In the curves, Fig. 14 distinctly shows this possibility in curve A, since the resistance reaches a large negative value at 2π and again at 4π. Likewise Fig. 11 shows the same possibility. On account of the resulting confusion in the figures, the higher frequency portions have not been drawn in the figures, but from (57), (59), (61), and (63) we can see what happens. Thus, for very high frequencies, η is large compared with unity, so that the formulas may be written,

$$R_0 = -\frac{12r_0}{\zeta^4}(\eta + \zeta)\sin\eta, \tag{57-a}$$

$$\begin{aligned} R_0 &= -\frac{12r_0}{\zeta^2}(1 + \cos\eta + \sin\eta), \\ &= -\frac{12r_0}{\zeta^2}\left[1 + \sqrt{2}\sin\left(\eta + \frac{\pi}{4}\right)\right], \end{aligned} \tag{59-a}$$

$$R_0 = -\frac{12r_0}{\xi^4}\left[1 - \frac{\xi}{\eta}\cos\xi + \frac{\xi}{\eta}\cos\eta\right], \tag{61-a}$$

$$R_0 = -\frac{12r_0}{\xi^2}. \tag{63-a}$$

It is noteworthy that all of these exhibit the possibility of "dwarf" waves separated by discrete frequency intervals except (63-a). On the other hand, (63-a) gives possible conditions for operation at all high frequencies provided that the proper external circuit may be secured.

VIII. Postscript

The extension of the electronics of vacuum tubes which was described in the preceding pages must be regarded in the light of a tenta-

tive starting point rather than as a completed structure. Of the fundamental correctness of the method of attack there can be little doubt. The various simplifying assumptions, however, require careful scrutiny and doubtless some of them will be revised as time goes on and additional experience is acquired. Experimental guidance will be invaluable, and indeed certain data already have been obtained which are helpful in analysis of the assumptions. Although these data are in general qualitative agreement with the theory as outlined, the experimental technique must be refined before quantitative comparison can be made. It is hoped that the results can be made available at an early date.

Among the various assumptions which were made in the development of the theory, there are three which lead particularly to far-reaching consequences. These three may be enumerated as follows:

1. Plane-parallel tube structures
2. Current flow in straight lines
3. Small alternating-current amplitudes.

There are grounds for the belief that the assumption of plane-parallel tube structures does not exclude the application of the alternating-current results to cylindrical structures as completely as might be supposed. In the first place, the approximation of cylindrical arrangements to the plane-parallel structure becomes better as the cathode diameter is made large. Many tubes contain special cathode structures where this is the case. Furthermore, Benham [1] has obtained an approximate solution for the alternating-current velocity in cylindrical diodes where the cathode diameter is vanishingly small, and the transit angle is less than 5 radians. The resulting curves of alternating-current velocity versus transit angle have the same shape as the curves for the planar structures, and when the cylindrical transit angle is arbitrarily increased by about 20 per cent, the quantitative agreement is fair for transit angles less than 4 radians. It follows that until accurate solutions for cylindrical triodes can be obtained, the planar solutions may be expected to give correct qualitative results, and fair quantitative results when appropriate modifications of the transit angle are made. In fact, good agreement is obtained if calculations of the cylindrical transit angle are made as though the structure were planar.

The assumption of current flow in straight lines is open to some question when a grid mesh is interposed in the current path. For the positive grid triode, the objection to the assumption has been overcome by postulating a special case where an ideal choke coil prevents

the grid from carrying away any of the alternating current. Benham [1] has suggested an alternative which seems to work fairly well when the grid-cathode path of a negative grid tube is considered, but which offers grave difficulties when the grid-plate path is included. A still different alternative was employed in the present paper in connection with negative grid triodes, and successfully indicates phase angles for the mutual conductance of the tube which are qualitatively logical. The grid-plate path is still without adequate treatment, however.

As to the third general assumption: that of relatively small alternating-current amplitudes, there can be no objection from a strictly mathematical point of view, and for a very large proportion of the physical applications the assumption is thoroughly justified. Indeed, it is the only one which is successful in giving *starting* conditions for oscillators. However, when questions as to the power efficiency of oscillators or amplifiers arise, then the "small signal" theory is inadequate, and should be supplanted by an approximate theory. The form which this approximate theory should take is indicated by the standard methods of dealing with the efficiencies of low-frequency power amplifiers and oscillators where the wave shape of the plate current is assumed to be given. The application of the same kind of approximation to ultra-high-frequency circuits may eventually prove to be a simpler matter than the "small signal" theory set forth in these pages.

Besides the three main assumptions discussed above, there was a fourth assumption which, although of lesser importance, deserves some comment. This fourth assumption involves the neglect of initial velocities at a hot cathode. If all electrons were emitted with the same velocity, the theory is adequate, and may be applied as indicated by Langmuir and Compton.[3] When the distribution of velocities according to Maxwellian, or Fermi-Dirac, laws is considered, some modifications may be necessary. In general, a kind of blurring of the clear-cut results of the univelocity theory may be expected, which will be expected to result in an increase in the resistive components of the various impedances at the expense of the reactive components. Again, lack of symmetry in the geometry of the tube structure may be expected to do the same thing, since the transit angles are then different in the different directions.

Finally, however, and with all its encumbering assumptions, it is hoped that the excursion back to fundamentals which was made in this paper, has resulted in a method of visualizing the motions of the condensations and rarefactions of the electron densities inside of vacuum tubes operating at high frequencies and has shown their relation to the conduction and displacement components of the total current.

Data for Fig. 11

ζ	$\eta=\zeta$		$\eta=\frac{1}{2}\zeta$		$\eta=\frac{1}{3}\zeta$		$\eta=\frac{1}{4}\zeta$	
	R_0	X_0	R_0	X_0	R_0	X_0	R_0	X_0
1.0	1.87	−0.577	0.302	−0.119			0.0633	
1.4	1.75	−0.777	0.292	−0.164			0.0604	−0.160
1.57	1.69	−0.855	0.288	−0.182	0.112	−0.172		
1.8	1.60	−0.949	0.280	−0.204	0.108	−0.193	0.0568	−0.200
2.356	1.36	−1.13	0.260	−0.241	0.0987	−0.240		
2.8	1.15	−1.23	0.241	−0.289			0.0458	−0.278
3.14	0.986	−1.27	0.226	−0.311	0.0838	−0.289	0.0418	−0.297
3.6	0.769	−1.29	0.204	−0.335	0.0748	−0.308	0.0364	−0.316
4.0	0.593	−1.27	0.186	−0.350	0.0673	−0.320	0.0320	−0.327
4.71	0.326	−1.18	0.153	−0.366	0.0457	−0.329		
5.2	0.186	−1.08	0.132	−0.368			0.0216	−0.330
5.6	0.0972	−0.987	0.116	−0.368			0.0193	−0.324
6.28	0	−0.830	0.0923	−0.358	0.0365	−0.309	0.0165	−0.306
6.8	−0.0348	−0.719	0.0767	−0.346			0.0153	−0.291
7.2	−0.0440	−0.644	0.0662	−0.337	0.0308	−0.284	0.0148	−0.278
7.85	−0.0369	−0.546	0.0515	−0.318	0.0284	−0.264		
8.4	−0.0199	−0.487	0.0414	−0.302	0.0268	−0.248	0.0144	−0.238
9.0	+0.000414	−0.444	0.0320	−0.284	0.0254	−0.233	0.0144	−0.222
9.42	0.0125	−0.424	0.0262	−0.272	0.0244	−0.222	0.0143	−0.235
10.0	0.0218	−0.407	0.0194	−0.256			0.0138	−0.198
10.99	0.0213	−0.385	0.0101	−0.232	0.0192	−0.193		
12.57	0	−0.342	0	−0.197	0.0128	−0.172	0.00962	−0.162

Data for Fig. 14

ζ	$\eta=\zeta$		$\eta=\frac{1}{2}\zeta$		$\eta=\frac{1}{3}\zeta$		$\eta=\frac{1}{4}\zeta$	
	R_0	X_0	R_0	X_0	R_0	X_0	R_0	X_0
1.0	2.94	−0.932	0.581	−0.225			0.133	−0.221
1.4	2.84	−1.33	0.595	−0.304			0.136	−0.286
1.57	2.78	−1.49	0.601	−0.336	0.252	−0.302		
1.8	2.67	−1.71	0.606	−0.377	0.255	−0.330	0.139	−0.336
2.356	2.30	−2.19	0.618	−0.465	0.264	−0.381		
2.8	1.90	−2.48	0.619	−0.528			0.147	−0.401
3.14	1.56	−2.63	0.613	−0.571	0.272	−0.424	0.150	−0.410
3.6	1.06	−2.71	0.598	−0.625	0.276	−0.438	0.153	−0.416
4.0	0.645	−2.67	0.577	−0.665	0.277	−0.448	0.155	−0.417
4.71	0.00015	−2.38	0.525	−0.728	0.275	−0.460	0.157	−0.412
5.2	−0.319	−2.08	0.479	−0.762			0.158	−0.407
5.6	−0.494	−1.80	0.436	−0.784			0.158	−0.404
6.28	−0.608	−1.31	0.356	−0.807	0.253	−0.476	0.156	−0.396
6.8	−0.565	−0.990	0.292	−0.814			0.154	−0.390
7.2	−0.480	−0.796	0.241	−0.803	0.232	−0.483	0.152	−0.386
7.85	−0.290	−0.594	0.160	−0.797	0.213	−0.485		
8.4	−0.111	−0.528	0.0957	−0.773	0.196	−0.486	0.144	−0.376
9.0	+0.00226	−0.536	0.0308	−0.735	0.175	−0.485	0.138	−0.372
9.42	0.0574	−0.574	−0.0102	−0.705	0.160	−0.484	0.133	−0.369
10.0	0.077	−0.634	−0.0581	−0.666			0.127	−0.365
10.99	0	−0.690	−0.118	−0.564	0.101	−0.436		
12.57	−0.152	−0.560	−0.152	−0.414	0.0445	−0.383	0.0938	−0.348

DATA FOR FIG. 15

ζ	$\eta = \zeta$		$\eta = \frac{1}{2}\zeta$		$\eta = \frac{1}{3}\zeta$		$\eta = \frac{1}{4}\zeta$	
	R_0	X_0	R_0	X_0	R_0	X_0	R_0	X_0
1.0			0.239	−3.06			0.179	−1.58
1.4			0.228	−2.22			0.172	−1.19
1.57			0.223	−2.00	0.199	−1.39		
1.8			0.215	−1.76	0.192	−1.24	0.162	−0.979
2.356			0.193	−1.39	0.173	−1.01		
2.8			0.173	−1.21			0.131	−0.737
3.14			0.157	−1.10	0.130	−0.825	0.120	−0.693
3.6			0.135	−0.983	0.123	−0.756	0.104	−0.645
4.0			0.116	−0.903			0.0902	−0.610
4.71			0.0837	−0.788	0.0796	−0.633		
5.2			0.0643	−0.724			0.0540	−0.524
5.6			0.0503	−0.677				
6.28			0.0308	−0.605	0.0346	−0.506	0.0308	−0.455
6.8			0.0197	−0.558			0.0232	−0.425
7.2			0.0131	−0.525	0.0194	−0.444	0.0187	−0.402
7.85			0.00568	−0.477	0.0126	−0.406		
8.4			0.00203	−0.442	0.00939	−0.378	0.109	−0.343
9.0			−0.0000284	−0.409	0.00710	−0.351	0.00908	−0.318
9.42			−0.000646	−0.388	0.00608	−0.333	0.00826	−0.290
10.0			−0.000817	−0.363			0.00744	−0.284
10.99			−0.000402	−0.327	0.00408	−0.282		
12.57			0	−0.285	0.00216	−0.246	0.00385	−0.227

DATA FOR FIG. 16

ζ	$\eta = \zeta$		$\eta = \frac{1}{2}\zeta$		$\eta = \frac{1}{3}\zeta$		$\eta = \frac{1}{4}\zeta$	
	R_0	X_0	R_0	X_0	R_0	X_0	R_0	X_0
1.0	−0.176	−9.35	0.426	−4.84			0.342	−2.52
1.4	−0.306	−6.84	0.366	−3.64			0.316	−1.96
1.57	−0.363	−6.15	0.338	−3.33	0.345	−2.32		
1.8	−0.436	−5.41	0.299	−3.00	0.321	−2.11	0.287	−1.67
2.356	−0.584	−4.15	0.206	−2.46	0.263	−1.77		
2.8	−0.658	−3.44	0.137	−2.17			0.212	−1.29
3.14	−0.685	−3.00	0.0888	−1.99	0.188	−1.48	0.190	−1.21
3.6	−0.685	−2.52	0.0318	−1.80	0.149	−1.36	0.162	−1.12
4.0	−0.661	−2.17	−0.0104	−1.65	0.120	−1.27	0.140	−1.06
4.71	−0.565	−1.69	−0.0698	−1.43	0.0747	−1.13	0.108	−0.957
5.2	−0.482	−1.45	−0.0998	−1.30			0.0871	−0.898
5.6	−0.413	−1.30	−0.119	−1.21			0.0730	−0.853
6.28	−0.304	−1.11	−0.141	−1.07	0.0478	−0.907	0.0523	−0.787
6.8	−0.236	−1.02	−0.151	−0.975			0.0389	−0.742
7.2	−0.195	−0.963	−0.155	−0.910	−0.0220	−0.805	0.0296	−0.709
7.85	−0.148	−0.894	−0.156	−0.815	−0.0364	−0.743		
8.4	−0.126	−0.848	−0.152	−0.747	−0.0458	−0.695	+0.0072	−0.625
9.0	−0.113	−0.803	−0.145	−0.680	−0.0538	−0.647	−0.00162	−0.589
9.42	−0.109	−0.771	−0.138	−0.638	−0.0582	−0.615	−0.00706	−0.565
10.0	−0.107	−0.727	−0.128	−0.588			−0.0135	−0.535
10.99	−0.101	−0.653	−0.107	−0.518	−0.0668	−0.517		
12.57	−0.0760	−0.556	−0.0760	−0.439	−0.0667	−0.439	−0.0314	−0.426